Solving a recurrence by expansion

Example 1 (corresponds to PrintXs.java)

$$T(n) = T(n-1) + 2 \quad , \quad T(1) = 3$$

Step 1: Calculate the first few values to get some insight and understanding

n	$T(n)$
1	3
2	5
3	7
4	9
5	11

Step 2: Expand the definition and look for a pattern

$$T(n) = T(n-2) + 2$$

$$= \left(T(n-3) + 2\right) + 2$$

$$= T(n-3) + 2 + 2$$

$$= \left(T(n-4) + 2\right) + 2 + 2$$

$$\vdots$$

$$= T(1) + \overbrace{2 + 2 + \cdots + 2}^{n-1 \text{ copies of } 2}$$

$$= 3 + 2(n-1)$$

$$= 2n + 1$$

Step 3: check agreement with step 1. Also use an online tool like Wolfram Alpha to check the answer.

Example 2 (corresponds to SayHi.java)

$$T(n) = T(n-3) + 2 \quad \text{for } n \geq 0 \quad ; \quad T(0) = T(1) = T(2) = 5.$$

Step 1: Calculate the first few values to get some insight and understanding

n	$T(n)$
0	5
1	5
2	5
3	7
4	7
5	7
6	9
7	9
8	9
9	11

Step 2: Expand the definition and look for a pattern

$$
\begin{aligned}
T(n) &= T(n-3) + 2 \\
&= T(n-6) + 2 + 2 \\
&= T(n-9) + 2 + 2 + 2 \\
&\ \ \vdots \\
&= T(0) + \underbrace{2 + 2 + 2 + \cdots + 2 + 2}_{n/3 \text{ copies of } 2} \\
&= 5 + 2 \cdot \frac{n}{3}
\end{aligned}
$$

Here we assumed n is a multiple of 3, for simplicity

So we conclude $T(n) = 5 + \frac{2n}{3}$, when n is a multiple of 3.

Step 3: check agreement with step 1. Also use an online tool like Wolfram Alpha to check the answer.

(but it may not help!!)

Example 3 -- more challenging, comes from DoManyIncrements.java:

$$T(n) = 3T(n-1) + 2, \qquad T(0) = 5$$

Step 1: Calculate the first few values to get some insight and understanding

n	$T(n)$
0	5
1	17
2	53
3	161
4	485
$\vdots$	$\vdots$

Step 2: Expand the definition and look for a pattern

$$T(n) = 3T(n-1) + 2$$

expand $= 3\left(3T(n-2)+2\right) + 2$

rewrite $= 3^2\, T(n-2) + 3\times 2 + 2$

expand $= 3^2\left(3T(n-3)+2\right) + 3\times 2 + 2$

rewrite $= 3^3\, T(n-3) + 3^2\times 2 + 3\times 2 + 2$

$$\vdots$$

notice the pattern $= 3^n\, T(n-n) + 3^{n-1}\times 2 + 3^{n-2}\times 2 + \cdots + 3\times 2 + 2$

rewrite $= 5\times 3^n + 2\times\left(\underbrace{3^{n-1} + 3^{n-2} + \cdots + 3^1 + 1}\right)$

Use formula for sum of geometric progression (or just use Wolfram Alpha): this is $(3^n - 1)/2$

rewrite $= 5\times 3^n + (3^n - 1)$

$$= 6\times 3^n - 1 = 2\times 3^{n+1} - 1$$

Step 3: check agreement with step 1. Also use an online tool like Wolfram Alpha to check the answer.